Lesson 4-3 Basic Trigonometric Identities

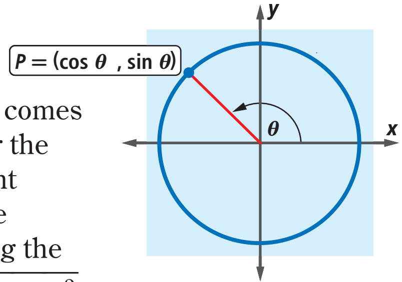
Vocabulary

identity

BIG IDEA If you know cos θ, you can easily find cos($-\theta$), cos($90° - \theta$), cos($180° - \theta$), and cos($180° + \theta$) without a calculator, and similarly for sin θ and tan θ.

An **identity** is an equation that is true for all values of the variables for which the expressions on each side are defined. There are five theorems in this lesson; all are identities.

The Pythagorean Identity



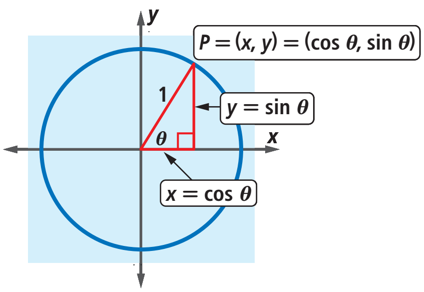
The first identity we derive in this lesson comes directly from the equation $x^2 + y^2 = 1$ for the unit circle. Because, for every θ, the point $P = (\cos\theta, \sin\theta)$ is on the unit circle, the distance from P to (0, 0) must be 1. Using the Distance Formula, $\sqrt{(\cos\theta - 0)^2 + (\sin\theta - 0)^2} = 1$. Squaring both sides of the equation gives $(\cos\theta)^2 + (\sin\theta)^2 = 1$. This argument proves a theorem called the *Pythagorean Identity*.

Pythagorean Identity Theorem

For every θ, $\cos^2\theta + \sin^2\theta = 1$.

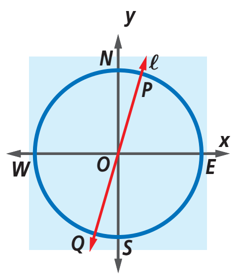
An abbreviated version of $(\cos\theta)^2$ is $\cos^2\theta$, the square of the cosine of θ. Similarly, $(\sin\theta)^2$ is written $\sin^2\theta$ and $(\tan\theta)^2$ is written $\tan^2\theta$. Notice that we do *not* write $\cos\theta^2$ for $(\cos\theta)^2$.

STOP **QY1**



The name of the above identity comes from the Pythagorean Theorem because in the first quadrant, as shown at the right, cos θ and sin θ are the sides of a right triangle with hypotenuse 1. Among other things, the Pythagorean Identity enables you to obtain either cos θ or sin θ if you know the other.

Mental Math



True or False

a. $\angle POE$ and $\angle POW$ are complementary.

b. $\angle POE$ and $\angle PON$ are supplementary.

c. $m\angle POE = m\angle QOW$

d. $m\angle POW = \pi - m\angle POE$

QY1

Which two expressions are equal?

A $\tan^2\theta$

B $\tan\theta^2$

C $(\tan\theta)^2$

Example 1

If $\cos\theta = \frac{3}{5}$, find $\sin\theta$.

Solution Substitute into the Pythagorean Identity.

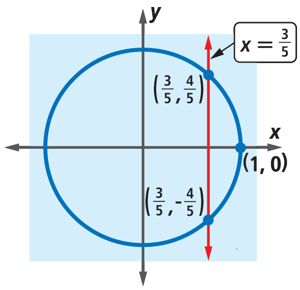
$$\left(\frac{3}{5}\right)^2 + \sin^2\theta = 1$$

$$\frac{9}{25} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{16}{25}$$

$$\sin\theta = \pm\frac{4}{5}$$

Thus, $\sin\theta = \frac{4}{5}$ or $\sin\theta = -\frac{4}{5}$.



Check Refer to the unit circle. The vertical line $x = \frac{3}{5}$ intersects the unit circle in two points. One is in the first quadrant, in which case the y-coordinate ($\sin\theta$) is $\frac{4}{5}$. The other is in the fourth quadrant, where $\sin\theta$ is $-\frac{4}{5}$.

STOP **QY2**

QY2

If $\sin\theta = 0.6$, what is $\cos\theta$?

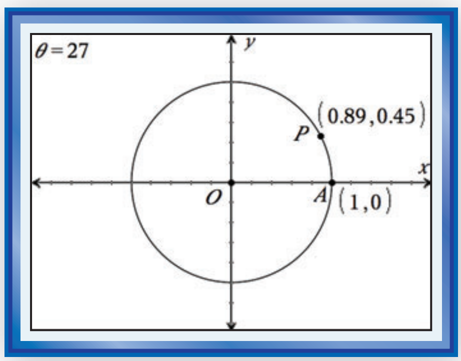
The Symmetry Identities

Many other properties of sines and cosines follow from their definitions and the symmetry of the unit circle. Recall that a circle is symmetric to any line through its center. This means that the reflection image of any point over one of these lines also lies on the circle.

Activity 1

MATERIALS DGS or graph paper, compass, and protractor

Step 1 Draw a unit circle on a coordinate grid. Plot the point $A = (1, 0)$. Pick a value of θ between 0° and 90°. Let a point P in the first quadrant be the image of A under the rotation R_θ. Find the values of $\cos\theta$ and $\sin\theta$ from the coordinates of P. A sample is shown at the right.



Step 2 Reflect P over the x-axis. Call its image Q. Notice that Q is the image of (1, 0) under a rotation of magnitude $-\theta$. Consequently, $Q = (\cos(-\theta), \sin(-\theta))$.

a. What are the values of $\cos(-\theta)$ and $\sin(-\theta)$ for your point Q?

b. How are $\cos\theta$ and $\cos(-\theta)$ related? What about $\sin\theta$ and $\sin(-\theta)$?

Step 3 Rotate your point P 180° around the circle. Call its image H. Notice that H is the image of (1, 0) under a rotation of magnitude $(180° + \theta)$. Consequently, $H = (\cos(180° + \theta), \sin(180° + \theta))$.

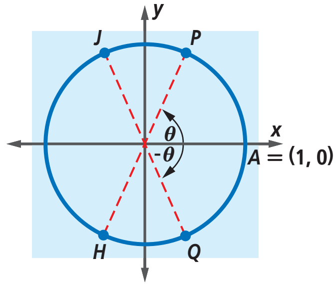
a. What are the values of $\cos(180° + \theta)$ and $\sin(180° + \theta)$ for your point H?

b. How are $\cos\theta$ and $\cos(180° + \theta)$ related? How are $\sin\theta$ and $\sin(180° + \theta)$ related?

Step 4 Use a calculator to find cos θ and sin θ for your value of θ in Step 1. Then find cos(–θ) and sin(–θ), and also cos(180° + θ) and sin(180° + θ). Explain any differences between the values displayed by the calculator and what you found in Steps 2 and 3.

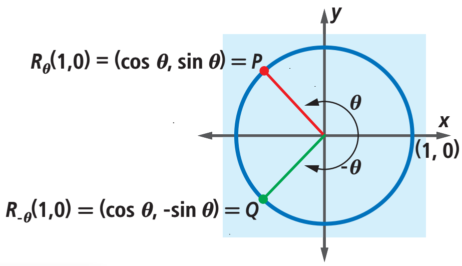
Save your work for Activity 2.

Activity 1 is based on the following ideas: When a point P on the unit circle is reflected over either axis, or when it is rotated through a half turn, either the coordinates of the three images are equal to the coordinates of P or they are opposites of the coodinates of P. The magnitudes of the rotations that map (1, 0) onto these points are θ (for P at the right), -θ (for Q), 180° + θ (for H), and 180° – θ (for J). So the sines and cosines of these magnitudes are either equal or opposites.



Sines and Cosines of Opposites

Rotations of magnitude θ and -θ go in opposite directions. The two rotation images are reflection images of each other over the x-axis. Thus they have the same first coordinates (cosines) but opposite second coordinates (sines). It follows that the ratios of the y-coordinates to the x-coordinate are opposites. This argument proves the following theorem.



Opposites Theorem

For all θ,

$\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$.

Sines and Cosines of θ + 180° or $\theta + \pi$

Adding 180° or π to the argument θ of a trigonometric function is equivalent to rotating halfway around the unit circle.

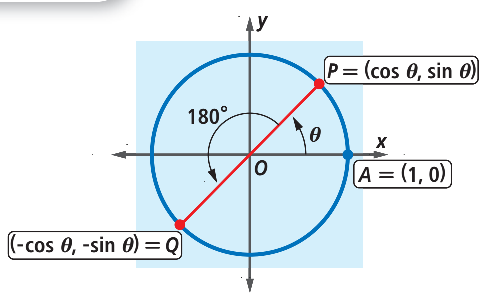
Half-Turn Theorem

For all θ, $\cos(180° + \theta) = -\cos\theta = \cos(\pi + \theta)$

$\sin(180° + \theta) = -\sin\theta = \sin(\pi + \theta)$

and $\tan(180° + \theta) = \tan\theta = \tan(\pi + \theta)$.

Proof Let $A = (1, 0)$ and let $P = R_\theta(A) = R_\theta(1, 0) = (\cos\theta, \sin\theta)$. Now let Q be the image of P under $R_{180°}$. Because $R_{180°}$ maps (a, b) to $(-a, -b)$, Q has coordinates $(-\cos\theta, -\sin\theta)$. But Q is also the image of A under a rotation of magnitude 180° + θ. So Q also has coordinates $(\cos(180° + \theta), \sin(180° + \theta))$. Equating the two ordered pairs for Q proves the first two parts of the theorem. The third part follows by dividing the second equation by the first.



Sines and Cosines of Supplements

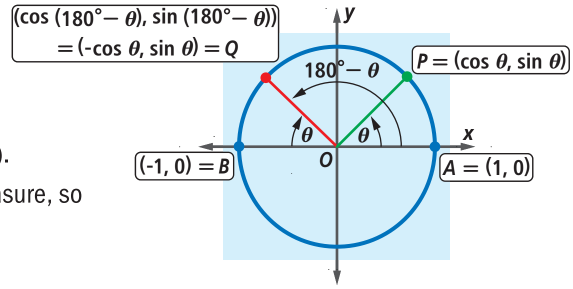
Recall that if an angle has measure θ, then its supplement has measure $180° - \theta$, that is, $\pi - \theta$. Activity 1 shows that the values of the trigonometric functions of θ and $180° - \theta$ are related, as stated in the following theorem.

Supplements Theorem

For all θ, $\sin(180° - \theta) = \sin\theta = \sin(\pi - \theta)$

$\cos(180° - \theta) = -\cos\theta = \cos(\pi - \theta)$

and $\tan(180° - \theta) = -\tan\theta = \tan(\pi - \theta)$.



Proof Let $P = (\cos\theta, \sin\theta)$. Let Q be the reflection image of P over the y-axis, as in the diagram at the right. Because the reflection image of (x, y) over the y-axis is $(-x, y)$,

$$Q = (-\cos\theta, \sin\theta).$$

Recall from geometry that reflections preserve angle measure, so

$$m\angle QOB = m\angle POA = \theta.$$

Also, since $\angle AOQ$ and $\angle QOB$ are a linear pair,

$$m\angle AOQ = 180° - \theta.$$

So, by the definitions of cosine and sine,

$$Q = (\cos(180° - \theta), \sin(180° - \theta)).$$

Thus, $(\cos(180° - \theta), \sin(180° - \theta)) = (-\cos\theta, \sin\theta)$.

The x-coordinates are equal, so

$$\cos(180° - \theta) = -\cos\theta.$$

Likewise, the y-coordinates are equal, so

$$\sin(180° - \theta) = \sin\theta.$$

Dividing the latter of these equations by the former gives the third part of the Supplements Theorem,

$$\tan(180° - \theta) = -\tan\theta.$$

STOP QY3

▶ QY3

Suppose $\sin\theta = 0.496$ and $\cos\theta = 0.868$. Without using a calculator, find

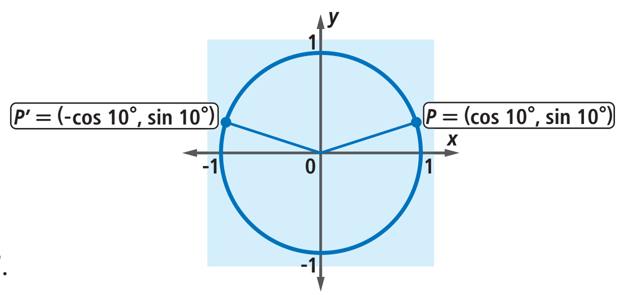
a. $\sin(\pi - \theta)$.

b. $\cos(180° - \theta)$.

Example 2

Given that $\sin 10° \approx 0.1736$, find a value of x other than $10°$ and between $0°$ and $360°$ for which $\sin x = 0.1736$.

Solution Think: $\sin 10°$ is the second coordinate of the image of $(1, 0)$ under $R_{10°}$. What other rotation will give the same second coordinate? It is the rotation that gives the reflection image of the point P in the diagram at the right. That rotation has magnitude $180° - 10°$, or $170°$. So $\sin 170° = \sin 10° = 0.1736$, and $x = 170°$.



If the requirement that $0° < x < 360°$ in Example 2 is relaxed, there are other answers. Because you can add or subtract 360° to the magnitude of any rotation and get the same rotation, $\sin 10° = \sin 170° = \sin 530° = \sin(-190°)$. Also, in radians, $\sin\left(\frac{\pi}{18}\right) = \sin\left(\frac{17\pi}{18}\right) = \sin\left(\frac{53\pi}{18}\right) = \sin\left(\frac{-19\pi}{18}\right)$.

STOP QY4

QY4

Given that $\cos 10° \approx 0.9848$, find a value of x other than 10° for which $\cos x = 0.9848$.

Sines and Cosines of Complements

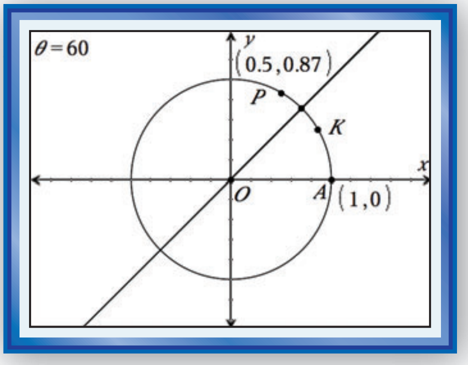
Activity 2

MATERIALS DGS or graph paper

Step 1 Begin with the graph from Step 3 of Activity 1. Hide points H and Q. Draw the line $y = x$. Again pick a value of θ between 0° and 90° and let $P = R_\theta(1, 0)$. Find $\cos\theta$ and $\sin\theta$ for your value of θ.

Step 2 Reflect point P over $y = x$ and call its image K. From your knowledge of reflections, what are the coordinates of K?

Step 3 In terms of θ, what is the magnitude of the rotation that maps $(1, 0)$ onto K? (Hint: K is as far from A along the circle as P is from the point $(0, 1)$.) Answer in both degrees and radians.



Step 4 Develop an identity that relates the sine and cosine of your answers to Step 3 to the sine and cosine of θ.

If an angle has measure θ, then its complement has measure $90° - \theta$ or $\frac{\pi}{2} - \theta$. Activity 2 shows that the sines and cosines of θ and $90° - \theta$ are related.

Complements Theorem

For all θ,

$$\sin(90° - \theta) = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

and $\cos(90° - \theta) = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

These theorems can help extend your knowledge of circular functions.

Example 3

Given that $\sin 30° = \frac{1}{2}$, compute the exact value of each function below.

a. $\cos 60°$ b. $\cos 30°$ c. $\sin 150°$

d. $\cos 210°$ e. $\sin(-30°)$

(continued on next page)

Solution

a. Use the Complements Theorem.
$\cos 60° = \sin(90° - 60°) = \sin 30°$. So $\cos 60° = \frac{1}{2}$.

b. Use the Pythagorean Identity Theorem. $\sin^2 30° + \cos^2 30° = 1$.
So $\cos^2 30° = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. Thus, $\cos 30° = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$.
However, we know $\cos 30°$ is positive, so $\cos 30° = \frac{\sqrt{3}}{2}$.

c. Use the Supplements Theorem. $\sin 150° = \sin(180° - 150°) = \sin 30°$. So $\sin 150° = \frac{1}{2}$.

d. Use the Half-Turn Theorem. $\cos 210° = \cos(180° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$.

e. Use the Opposites Theorem. $\sin(-30°) = -\sin 30° = -\frac{1}{2}$.

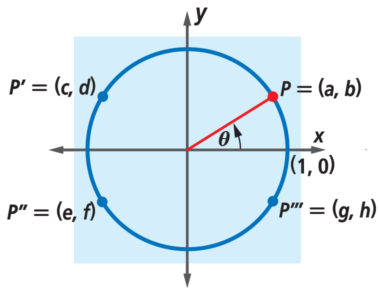
In using these identities, you should also be able to use the unit circle to do a visual check of your answers or to derive a property if you forget one.

Questions

COVERING THE IDEAS

1. **True or False** When $\theta = 180°$, $\cos^2\theta + \sin^2\theta = 1$.
2. a. If $\sin\theta = \frac{24}{25}$, what are two possible values of $\cos\theta$?
   b. Draw a picture to justify your answers to Part a.
3. If $\tan\theta = 3$, what is $\tan(-\theta)$?
4. a. **True or False** $\cos 14° = \cos(-14°)$
   b. Justify your answer to Part a with a unit circle diagram.

In 5 and 6, refer to the figure at the right. $P = R_\theta(1, 0)$, $P' = r_{y\text{-axis}}(P)$, $P'' = R_{180°}(P)$, and $P''' = r_{x\text{-axis}}(P)$.



5. Which coordinates equal $\cos(180° - \theta)$?
6. Which coordinates equal $\sin(180° + \theta)$?
7. **True or False** $\sin(-\theta) = \sin\theta$

In 8 and 9, $\sin\theta = \frac{1}{3}$. Evaluate without using a calculator.

8. $\sin(-\theta)$
9. $\sin(180° - \theta)$
10. Using what you know about $\sin(180° - \theta)$ and $\cos(180° - \theta)$, explain why $\tan(180° - \theta) = -\tan\theta$.
11. Use a calculator to verify the three parts of the Supplements Theorem when $\theta = 146.5°$.

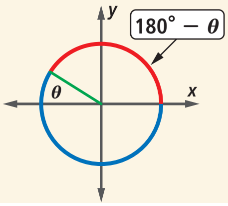
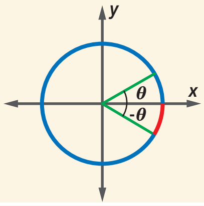
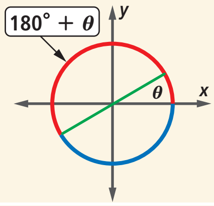
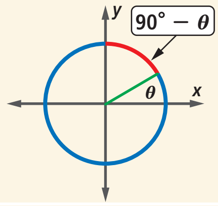
In 12 and 13, suppose $\cos x = \frac{5}{13}$. Evaluate without using a calculator.

12. $\cos(180° + x)$
13. $\sin(90° - x)$

In 14 and 15, $\tan y = k$. Evaluate.

14. $\tan(-y)$
15. $\tan(180° - y)$

16. Copy the table below, filling in the blank entries and completing the diagrams, to summarize the theorems in this lesson.

x	$180° - \theta$	$-\theta$	$180° + \theta$	$90° - \theta$
Diagram				
cos x	$-\cos\theta$	?	?	?
sin x	?	$-\sin\theta$	?	?
tan x	$-\tan\theta$	?	$\tan\theta$	?

APPLYING THE MATHEMATICS

In 17–21, the display below shows inputs and outputs of a CAS in degree mode. What theorem justifies each statement?

17.	cos(350)	cos(10)
18.	sin(160)	sin(20)
19.	sin(202)	-sin(22)
20.	cos(84)	sin(6)
21.	tan(187)	tan(7)

22. Prove that $\sin(\pi - \theta) = \sin\pi - \sin\theta$ is *not* an identity.

In 23–26, from the fact that $\sin 18° = \frac{\sqrt{5}-1}{4}$, find each value.

23. $\sin 162°$ 24. $\sin(-18°)$ 25. $\sin\frac{11\pi}{10}$ 26. $\cos\frac{2\pi}{5}$

REVIEW

In 27–29, without using a calculator, give exact values. (Lesson 4-2)

27. $\sin 90°$ 28. $\cos 810°$ 29. $\tan(90° + 90°)$

30. Convert $\frac{11}{6}$ clockwise revolutions to degrees. **(Lesson 4-1)**

31. a. What is the magnitude of the rotation of the minute hand of a clock in 6 minutes?
b. What is the measure of the angle between the minute hand and the second hand of a clock at exactly 12:06 A.M.? **(Lesson 4-1)**

32. Find an equation for the image of the graph of $y = x^2$ under the scale change $(x, y) \rightarrow \left(\frac{1}{2}x, 5y\right)$. **(Lesson 3-5)**

EXPLORATION

33. Use a calculator to investigate whether $\frac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$ is an identity. Try to prove your conclusion, either by providing a counterexample or by using definitions and properties.

QY ANSWERS

1. A and C
2. $\cos\theta = \pm\sqrt{1 - 0.6^2}$
$= \pm 0.8$
3. **a.** 0.496 **b.** –0.868
4. Answers vary. Samples: –10°, 370°, –350°